A generalisation of the Buchdahl transformation and perfect fluid solutions

1982 J. Phys. A: Math. Gen. 15 1799

(http://iopscience.iop.org/0305-4470/15/6/018)

J. Phys. A: Math. Gen. 15 (1982) 1799–1805. Printed in Great Britain

A generalisation of the Buchdahl transformation and perfect fluid solutions

B W Stewart†

Department of Physics, University of Cincinnati, Cincinnati, Ohio 45221, USA

Received 26 August 1981, in final form 21 January 1982

Abstract. We examine perfect fluid solutions in general relativity. The pressure isotropy relation possesses a discrete symmetry in the derivatives of the metric functions. It is shown that a generalisation of Buchdahl's original transformation allows new physically reasonable solutions to be obtained from known solutions. An example is given. Previously noticed first by Buchdahl, and later by Glass and Goldman, this symmetry was thought to lead only to unphysical solutions.

1. Introduction

Several authors have examined the problem of obtaining static, spherically symmetric, perfect fluid interior solutions in general relativity. Even under these assumptions, analytic and physically reasonable solutions are difficult to come by. On the other hand, numerical techniques can be applied to obtain physically motivated solutions. These solutions are not always amenable to easy analysis, however.

Some years ago, Buchdahl (1956) discovered that the Einstein equations for a perfect fluid possessed a discrete symmetry. This symmetry allowed a new solution to be obtained from a seed solution by a transformation of the time-like metric function (the Buchdahl transformation). This transformation could be applied to any static perfect fluid solution. No differential equations had to be solved. Later Glass and Goldman (1978) also noticed the symmetry, but concluded as did Buchdahl that only unphysical new solutions resulted through its application.

In this paper, we re-examine the use of the Buchdahl transformation (BT) and find that, if it is suitably generalised, it is possible to obtain physically reasonable new solutions through its application. We discuss the BT as applied to static, spherically symmetric, perfect fluid solutions. Since it is possible that *all* regular static perfect fluid solutions which can be matched to asymptotically flat vacuum solutions are also spherically symmetric, our restriction is not too severe.

2. The Buchdahl transformation

Buchdahl (1956) found that, given a static solution to the Einstein field equations for a perfect fluid

$$ds^2 = g_{00}\, dt^2 + g_{ij}\, dx^i\, dx^j, \tag{1}$$

† Present address: Department of Physics, Thomas More College, Fort Mitchell, Kentucky, USA.

the reciprocal line element, given by

$$ds^2 = (g_{00})^{-1}\, dt^2 + (g_{00})^2 g_{ij}\, dx^i\, dx^j, \tag{2}$$

was also a solution to the Einstein field equations for a perfect fluid.

If the mass density, ρ_0, and the pressure, p_0, are the quantities associated with the solution (1), then Buchdahl calculated the mass density and the pressure of the solution (2) to be

$$p = (g_{00})^{-2} p_0, \qquad \rho = -(g_{00})^{-2}(\rho_0 + 6p_0). \tag{3}$$

Since it is generally believed that the mass density must be positive in order for a solution to describe a physically reasonable situation, and since perfect fluids are expected to exhibit only pressure and not tension, ρ_0 and p_0 will be positive if the solution (1) is to be physically relevant. Therefore, Buchdahl concluded from (3) that the mass density of the solution (2) is negative, eliminating (2) as a physically reasonable solution. It seemed that, given a physically reasonable solution (in the sense discussed above), the BT produced only an unphysical solution.

If we intend to apply the BT in order to obtain new perfect fluid solutions appropriate for the interiors of any finite body (a neutron star, for example), it is clear that we must impose some boundary conditions on the metric. In § 3, we indicate one method (although not Buchdahl's original approach) of 'deriving' the BT. We also demonstrate the impossibility of using the BT to obtain a perfect fluid satisfying the Lichnerowicz (1955) boundary conditions from another perfect fluid solution which also satisfies the Lichnerowicz boundary conditions. We find that the BT can be modified so that it becomes possible to obtain a new perfect fluid solution appropriate for the description of a finite body from a known solution of the same type. In addition, this modification allows one to obtain physically reasonable ($\rho > 0, p \geqslant 0$) new solutions from known physically reasonable solutions.

3. Field equations

We begin by examining the non-vacuum, spherically symmetric, static Einstein equations for the line element of the form

$$ds^2 = e^{\nu}\, dt^2 - (1+\Phi)^4\, d\sigma^2, \tag{4}$$

$$d\sigma^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{5}$$

where ν, Φ are functions of r only. In order that the interior metric joins properly (Lichnerowicz 1955) to the exterior Schwarzschild solution†

$$ds^2 = \left(\frac{1 - M/2r}{1 + M/2r}\right)^2 dt^2 - \left(1 + \frac{M}{2r}\right)^4 d\sigma^2 \tag{6}$$

at the boundary of the body, $r = a$, we will require ($' \equiv d/dr$)

$$\begin{aligned} &\Phi(a) = M/2a, \qquad \Phi'(a) = -M/2a^2, \\ &\nu(a) = 2\ln(1 - M/2a) - 2\ln(1 + M/2a), \\ &\nu'(a) = (2M/a^2)(1 - M^2/4a^2)^{-1}. \end{aligned} \tag{7}$$

M is the mass of the body as measured by a distant observer.

† For a recent discussion of boundary conditions in general relativity see Bonnor and Vickers (1981).

The field equations are (Tolman 1962) (units $8\pi G = c = 1$)

$$T_0^0 = -4(1+\Phi)^{-5}(\Phi''+2\Phi'/r), \tag{8a}$$

$$-T_1^1 = (1+\Phi)^{-4}\left(\frac{2\Phi'\nu'}{1+\Phi}+\frac{\nu'}{r}+\frac{4\Phi'^2}{(1+\Phi)^2}+\frac{4\Phi'}{r(1+\Phi)}\right), \tag{8b}$$

$$-T_2^2 = -T_3^3 = (1+\Phi)^{-4}\left(\frac{2\Phi''}{1+\Phi}-\frac{2\Phi'^2}{(1+\Phi)^2}+\frac{2\Phi'}{r(1+\Phi)}+\frac{\nu''}{2}+\frac{\nu'^2}{4}+\frac{\nu'}{2r}\right). \tag{8c}$$

The energy–momentum tensor of a perfect fluid in this case can be written as

$$T_0^0 = \rho, \tag{9a}$$

$$-T_1^1 = -T_2^2 = -T_3^3 = p, \tag{9b}$$

$$T_j^i = 0 \qquad \text{for } i \neq j, \tag{9c}$$

where ρ is the mass density and p is the pressure.

From the isotropy of pressure relation (9*b*), and equations (8*b*) and (8*c*), we have

$$\frac{\nu''}{2}+\frac{\nu'^2}{4}-\frac{\nu'}{2r}-\frac{2\Phi'\nu'}{1+\Phi} = -\frac{2\Phi''}{1+\Phi}+\frac{6\Phi'^2}{(1+\Phi)^2}+\frac{2\Phi'}{r(1+\Phi)}. \tag{10}$$

We will now show that equation (10) is invariant under a discrete symmetry; i.e., under a certain transformation of ν and Φ the isotropy of pressure, equation (10), is retained.

If we now make a change of dependent variables

$$\tfrac{1}{2}\nu' = \tfrac{1}{2}(T+S), \tag{11}$$

$$2\Phi'/(1+\Phi) = \tfrac{1}{2}(T-S), \tag{12}$$

equation (10) becomes

$$T' - T/r - \tfrac{1}{2}T^2 + ST + \tfrac{1}{2}S^2 = 0. \tag{13}$$

It can be seen immediately that although equation (13) is a Riccati equation in T, it is an algebraic (quadratic) equation in S. Therefore, there will exist two solutions S_1, S_2 for every T. Suppose that (S_1, T) is a known solution to Einstein's equations for a perfect fluid. Then (S_2, T) is another perfect fluid solution where

$$S_2 = -(S_1 + 2T). \tag{14}$$

In order to see the equivalence of equation (14) and Buchdahl's result for equations (1), (2) and (3), we substitute equations (11), (12) into equation (14) and integrate. We obtain

$$(1+\Phi)^4 = (1+\Phi_0)^4\, e^{2\nu_0}, \tag{15}$$

$$e^{\nu} = e^{-\nu_0}, \tag{16}$$

where ν_0, Φ_0 are the 'metric functions' of the old (S_1, T) solution. Thus the result obtained here is equivalent to the Buchdahl result. Another method of obtaining the BT using the invariance of the Lagrangian is given in the Appendix.

Suppose that the spherically symmetric line element

$$ds^2 = A\, dt^2 - B\, d\sigma^2 \tag{17}$$

is a solution to the Einstein equations for a perfect fluid. Further suppose that at some finite radius, $r=a$, the interior solution (17) is to be matched to the vacuum Schwarzschild solution

$$ds^2=\left(\frac{1-M/2r}{1+M/2r}\right)^2 dt^2-\left(1+\frac{M}{2r}\right)^4 d\sigma^2. \tag{18}$$

The Lichnerowicz boundary conditions applied to the time-like component of the metric tensor are

$$A(r=a)=\left(\frac{1-M/2a}{1+M/2a}\right)^2, \tag{19a}$$

$$\left.\frac{dA}{dr}\right|_{r=a}=\frac{2M}{a^2}\left(1-\frac{M}{2a}\right)\left(1+\frac{M}{2a}\right)^{-3}. \tag{19b}$$

Next let us apply the BT to the solution (17). Application results in the following perfect fluid solution

$$ds^2=A^{-1} dt^2-A^2B\, d\sigma^2. \tag{20}$$

The Lichnerowicz boundary conditions applied to the time-like component of the metric tensor of the new solution are

$$A(r=a)=\left(\frac{1+M/2a}{1-M/2a}\right)^2, \tag{21a}$$

$$\left.\frac{dA}{dr}\right|_{r=a}=-\frac{2M}{a^2}\left(1+\frac{M}{2a}\right)\left(1-\frac{M}{2a}\right)^{-3}. \tag{21b}$$

It is evident that only if $M=0$ can the same function, A, satisfy both sets of boundary conditions. This, of course, represents Minkowski space–time and is of no interest here.

Since the field equations are second-order differential equations, the solutions ν_0, Φ_0 will in general contain two arbitrary constants. These constants are determined only after application of the boundary conditions. The transformation (14) preserves the functional form of the ν_0, Φ_0 functions. However, the values of the arbitrary constants will not remain the same. These (different) constants will be determined only after applicaton of the boundary conditions to the new solution (15) and (16). To ensure that this point is clear, we will partially work through an example.

In a perfect fluid solution given recently by Bayin (1978) the line element is

$$ds^2=(Ar^2+B)^2 dt^2-(Cr^2+D)^{-6} d\sigma^2 \tag{22}$$

where A, B, C, D are constants.

Application of the boundary conditions to the time-like component of the metric tensor yields

$$A=(M/2a^3)(1-2M/a)^{-1/2}, \tag{23a}$$

$$B=(1-5M/2a)(1-2M/a)^{-1/2}. \tag{23b}$$

After imposing the BT upon the solution (22)

$$g_{00}=(Ar^2+B)^{-2}. \tag{24}$$

Again, applying the boundary conditions gives

$$A = -(M/2a^3)(1-2M/a)^{-3/2}, \tag{25a}$$

$$B = (1-3M/2a)(1-2M/a)^{-3/2}. \tag{25b}$$

Particularly important is the change in sign in the constant A.

Since, as has been shown above, some of the constants involved in ρ_0 and p_0 will almost certainly change sign, it is possible that ρ_0 and p_0 will change sign also. This would allow ρ and p in equation (3) to be positive. In § 4 we give such an example.

4. Example

As an example, we will apply the method to the incompressible fluid Schwarzschild solution. In this case (C_1, C_2, A are constants)

$$S_1 = \frac{2C_1 r}{C_1 r^2 + C_2}, \qquad T = \frac{2C_1 r}{C_1 r^2 + C_2} - \frac{4r}{r^2 + A}. \tag{26}$$

Using equations (15) and (16) and applying the boundary condition (7)

$$(1+\Phi)^4 = \left(1-\frac{M}{2a}\right)^2 \left(1-\frac{Mr^2}{2a^3}\right)^{-6} \left(1+\frac{M}{a}-\frac{Mr^2}{a^3}-\frac{M^2r^2}{4a^4}\right)^4, \tag{27}$$

$$e^{\nu} = \left(1-\frac{M}{2a}\right)^2 \left(1-\frac{Mr^2}{2a^3}\right)^2 \left(1+\frac{M}{a}-\frac{Mr^2}{a^3}-\frac{M^2r^2}{4a^4}\right)^{-2}. \tag{28}$$

The components of the energy–momentum tensor are

$$\rho = \frac{6M}{a^3}\left(1-\frac{M}{2a}\right)^{-2}\left(1-\frac{Mr^2}{2a^3}\right)^4 \frac{(1+2Mr^2/a^3-2M/a-M^2r^2/4a^4)}{(1+M/a-Mr^2/a^3-M^2r^2/4a^4)^5}, \tag{29}$$

$$p = \frac{3M^2}{a^6}\left(1-\frac{M}{2a}\right)^{-2}\left(1-\frac{Mr^2}{2a^3}\right)^4 \frac{(a^2-r^2)}{(1+M/a-Mr^2/a^3-M^2r^2/4a^4)^5}. \tag{30}$$

Upon examination of the expressions (27), (28), (29) and (30), the example solution is found to exhibit the following properties.

(1) The time-like component of the metric tensor, e^{ν}, is zero if $M/a = 2$. This would give rise to an event horizon.

(2) Likewise, the space-like component of the metric tensor, $(1+\Phi)^4$, is zero if $M/a = 2$.

(3) The mass density, ρ, is non-negative at the centre if $M/a \leqslant \frac{1}{2}$. ρ is singular if $M/a = 2$. Also $\rho' \geqslant 0$. Thus the density is an increasing function of r. As a consequence, if $\rho \geqslant 0$ at the centre, it will be non-negative everywhere. The requirement that $\rho \geqslant 0$ is called the *weak energy condition*. The fact that $\rho \geqslant 0$ almost certainly means that the solution is unstable under perturbation.

(4) $p(r=a) = 0$. Thus the body has a finite boundary without the introduction of surface shells. p is singular if $M/a = 2$. p is a decreasing function of r. $p \geqslant 0$ throughout, also, at least as long as $M/a < 2$.

(5) $\rho \geqslant p$ everywhere in the interior if $M/a \leqslant \frac{2}{5}$. Since in addition for $M/a \leqslant \frac{2}{5}$ both ρ and p are positive, the *dominant energy condition* is satisfied for these values of M/a.

(6) Another energy condition, the *trace condition* $T^{\mu}_{\mu} = \rho - 3p \geqslant 0$, is satisfied if $M/a \leqslant \frac{2}{7}$. The trace condition is evidently more stringent than either the weak or the dominant energy conditions.

It would seem that the only unphysical property of the example solution is that the mass density is not a decreasing function of the radial coordinate. This may be due to the choice of the constant density solution as a seed solution.

5. Discussion

Due to their nonlinearity, Einstein's equations in the presence of matter have yielded only a handful of analytic, physically reasonable solutions. Even the simplifying assumptions of spherical symmetry and static perfect fluid sources have not increased the tractability of the equations significantly. Using the Buchdahl transformation as described in this paper, it is possible to obtain a new solution for every known perfect fluid solution. The resulting new solutions should be useful in gaining greater insight into the theory.

Acknowledgments

I would like to thank Drs Louis Witten, F Paul Esposito and Demetrios Papadapoulos for several beneficial conversations during the preparation of this manuscript. The referees of this paper have also made several useful recommendations.

Appendix

Perhaps a more elegant manner of obtaining the Buchdahl transformation is presented here†. Given the line element

$$ds^2 = -F\,dt^2 + F^{-1}\overline{g}_{ij}\,dx^i\,dx^j \tag{A1}$$

the Lagrangian density for a perfect fluid is

$$\mathscr{L} = \sqrt{\bar{g}}(\bar{R} + F_{,i}F^{,i}/2F^2 - 2p/F), \qquad {}_{,i} = \partial/\partial x^i, \tag{A2}$$

where the bars denote quantities associated with the three-metric $\overline{g}_{ij}$. $\bar{R}$ is the Ricci scalar of this three-metric, $\overline{g}_{ij}$, and $\bar{g}$ is the determinant. The Lagrangian density is form invariant under the transformations

$$F \to F^{-1}, \qquad p \to F^{-2}p, \tag{A3}$$

as can be readily verified. Since the field equations are obtained from the Lagrangian via application of a variational principle, the field equations will also be form invariant under the transformation (A3). The *forms* of the solutions of these field equations will then also remain unchanged.

† For a more detailed discussion of Lagrangian invariance transformations see Kramer *et al* (1980).

References

Bayin S S 1978 *Phys. Rev.* D **18** 2745
Bonnor W B and Vickers P A 1981 *Gen. Rel. Grav.* **13** 29
Buchdahl H A 1956 *Aust. J. Phys.* **9** 13
Glass E N and Goldman S P 1978 *J Math. Phys.* **19** 856
Kramer D, Stephani H, MacCallum M and Herlt E 1980 *Exact Solutions of Einstein's Field Equations* (Cambridge: CUP) p 339
Lichnerowicz A 1955 *Theories Relativistes de la Gravitation et de l'Electromagnetisme* (Paris: Masson et Cie) p 5
Tolman R C 1962 *Relativity, Thermodynamics, and Cosmology* (Oxford: Clarendon) p 245